

Fluid limits of many-server queues with state dependent service rates

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Abstract

We study a many-server queuing system with general service time distribution and state dependent service rates. The dynamics of the system are modeled using measure valued processes which keep track of the residual service times. Under suitable conditions, we prove the existence of a unique fluid limit.

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1 Introduction

In recent days, many-server queuing systems have received much attention due to its applications to call center. Thus it has become important to study its asymptotic properties to gain insight into the behavior of these systems. Studying different scaling limits (fluid or diffusion scaling) are established tradition in queuing theory. In the celebrated work of Halfin and Whitt [7], it was shown that for Poissonian arrival and exponential service time, with positive probability, there is a positive queue in the asymptotic regime if the arrival rate λ_n and the number of servers n both goes to infinity in a manner that $\lambda_n = n - \beta\sqrt{n}$ for some $\beta > 0$. Fluid and diffusion limits for the total number of customers in a network with time varying Poissonian arrival and staffing was obtained in [12]. This work was generalized in [11] for $G_t/M_t/s_t$ queuing networks with abandonment where the authors studied long-time behavior of the fluid limits.

A recent statistical study by Brown et. al. [2] suggests that it may be more appropriate to consider non-exponential service times. In particular, it is log-normal in some cases as shown in [2]. This emphasizes the need to consider many-server model with generally distributed service times. In [15], Whitt considered G/G/n network with abandonment and proposed a deterministic fluid approximation. The author proved convergence for discrete time model. In [10], Kaspi and Ramanan considered G/G/n model and obtained a measure-space valued fluid

limit. Later Kang and Ramanan generalized this work by allowing the customers abandonment in [8]. In [1], Atar et al. studied multi-class many-server queues with fixed priority and established the existence of unique fluid limits. Kang and Ramanan studied ergodic properties of the scaled processes for GI/G/n+G model and its relation with invariant states of the fluid limit in [9]. Reed in [14], established the fluid and diffusion limits of the customer-count processes for many-server queuing system under the finite first moment assumption on service time distribution. In [17], Zhang obtained the fluid limits for GI/G/n+G queuing systems.

All of the above models consider servers that serve the customers at a constant rate 1. In this work, we allow the servers to adjust their service rate depending on the number of customers in the system (equivalently, the number of customers in the queue). It is often useful to increase the service rates when the queue length is large. Management may also be interested to adjust the service rate depending on customers feedback. State dependent arrival and service rate were first introduced in [16] for conventional heavy traffic approximations. In case of single server models, processor sharing model is an example where service rate at any instant of time depends on the number of customers in the system. For some recent developments on processor sharing, we refer [6], [18], [13].

In this work, we consider a system with n -homogeneous servers. Customers arrival is given by a renewal process and customers are served under FCFS policy. Arrived customers do not leave the system until served. Let X_t^n denote the number of customers in the n -th system at time $t \geq 0$. Define $\bar{X}_t^n = \frac{1}{n}X_t^n$. The service rate of each server at time t is given by $k^n(\bar{X}_t^n)$ for some bounded map k^n on $[0, \infty)$. Note that, $k^n(\bar{X}_t^n)$ could be 0 for non-zero \bar{X}_t^n . The system is described by (Q_t^n, \mathcal{Z}_t^n) where Q_t^n denotes the number of customers waiting in the queue at time t and \mathcal{Z}_t^n is a non-negative Borel measure on $(0, \infty)$ such that $\mathcal{Z}_t^n(C)$ denotes the number of customers in service with their remaining service requirements in C , for $C \in \mathcal{B}((0, \infty))$. Thus \mathcal{Z}^n is a measure-valued process that keeps track of customers remaining service requirements. Measure-valued processes that keep track of residual service requirements of individual customers, have been considered earlier in literature (see [4], [6], [18], [17]). In [10], the authors used measure-valued processes that keep track of the time spent by the individual customers in service. Also their proof relies on the fact that there exists a compensator for the departure processes (see Corollary 5.5 there). Since we are allowing our service rate to be dynamic depending on \bar{X}^n , getting an explicit compensator for analogous processes as in [10] is a hard problem. In this work, we closely follow the approach in [18] (see also [17]). We show that the fluid limit of $\frac{1}{n}\mathcal{Z}_t^n$ is uniquely determined by an integral equation, referred as fluid model equation((9) below). A similar type of equation was also obtained in [18].

This paper is organized as follows. In Section 2, we introduce our GI/G/n model with our basic assumptions and state our main result. The uniqueness part of the main result is done in Section 2.1. In Section 3, we prove relative compactness of the stochastic pre-limits and characterizations of the limits are done in Section 4. Finally, in Appendix we prove existence result for solution to certain integral equation and recall some results from [18] those are used in this paper.

1.1 Notations

The following notations will be used throughout this paper. By \mathbb{N}, \mathbb{R} , we denote the set of natural numbers and the set of real numbers, respectively. Given $a, b \in \mathbb{R}$, the maximum (minimum) is denoted by $a \vee b (a \wedge b)$. We use a^+ for $a \vee 0$. We define $\mathbb{R}_+ = [0, \infty)$. For any $A \subset [0, \infty)$, we define $A^\epsilon = \{x \geq 0 : \inf_{a \in A} |x - a| < \epsilon\}$. For any $x \in \mathbb{R}$, the sets $(x, \infty), [x, \infty)$ will be denoted by C_x, \bar{C}_x respectively. For any topological space \mathcal{S} , $C_b(\mathcal{S})$ denotes the set of all real valued bounded, continuous map on \mathcal{S} and $\mathcal{B}(\mathcal{S})$ is used to denote the Borel σ -field of \mathcal{S} . $C([a, b], \mathcal{S})$ will denote the set of all continuous function from $[a, b]$ to \mathcal{S} . For any $f \in C([a, b], \mathbb{R})$, we define $|f|_{ab} = \sup_{x \in [a, b]} |f(x)|$.

The set of all non-negative finite Borel measures on $[0, \infty)$ is denoted by \mathcal{M} and \mathcal{M}_+ denotes the subset of \mathcal{M} containing all the measures having no atom at $\{0\}$. For any $\mu \in \mathcal{M}$ and Borel measurable function g on $[0, \infty)$, we define $\langle g, \mu \rangle = \int g d\mu$. For $\mu_1, \mu_2 \in \mathcal{M}$, the Prohorov metric is defined by

$$\rho(\mu_1, \mu_2) = \inf\{\epsilon > 0 : \mu_1(A) \leq \mu_2(A^\epsilon) + \epsilon, \mu_2(A) \leq \mu_1(A^\epsilon) + \epsilon, \text{ for all closed } A \subset [0, \infty)\}.$$

It is well known that (\mathcal{M}, ρ) is a Polish space (see Appendix in [3]). Also this topology is equivalent to the weak topology on \mathcal{M} which is characterized as follows: $\mu_n \rightarrow \mu$ in weak topology if and only if

$$\langle f, \mu_n \rangle \rightarrow \langle f, \mu \rangle \text{ for all } f \in C_b([0, \infty)).$$

Given any Polish space (E, π) , $D([0, \infty), E)$ denote the space of functions that are right-continuous with finite left limits (RCLL). Endow the space $D([0, \infty), E)$ with the Skorohod-Prohorov-Lindvall metric or J_1 metric, defined as

$$d(\phi, \phi') = \inf_{f \in \Upsilon} \left(\|f\|^\circ \vee \int_0^\infty e^{-u} d_u(\phi, \phi', f) du \right), \quad \phi, \phi' \in D([0, \infty), E)$$

where

$$d_u(\phi, \phi', f) = \sup_{t \geq 0} [\pi(\phi(t \wedge u), \phi'(f(t) \wedge u)) \wedge 1],$$

and Υ is the set of strictly increasing, Lipschitz continuous functions from $[0, \infty)$ onto itself, with

$$\|f\|^\circ = \sup_{0 \leq s < t} \left| \log \frac{f(t) - f(s)}{t - s} \right| < \infty.$$

As is well known [5], $D([0, \infty), E)$ is a Polish space under the induced topology.

We use " \Rightarrow " to denote the convergence in the sense of distribution.

2 Queuing model

In this section, we describe our GI/G/n model and the measure valued state descriptors. We assume that for each n , all the stochastic variables below, are defined on probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$. The system contains n identical servers. Each arriving customer has a single service requirement and is served by a single server. The customers are served by FCFS policy and they leave the system once their service is completed. We do not allow the customers to renege the system until their job is done. We also assume that the system works under work conserving policy i.e., all the servers are busy if there is a queue. We assume the following:

- Customers arrive according to a renewal process E_t^n with mean inter-arrival time $\frac{1}{\lambda^n}$ for some $\lambda^n > 0$.
- Service requirement of the arriving i -th customer is given by v_i^n where $\{v_i^n\}_{i=-\infty}^{i=\infty}$ is an positive valued i.i.d. sequence with common distribution ν^n .

Upon arrival the customers join the queue if all the servers are busy. At time t , all the servers serve the customers at a rate $k^n(\bar{X}_t^n)$ where $\bar{X}_t^n = \frac{X_t^n}{n}$ and X_t^n denotes the number of customers in system at time t . It is reasonable to assume that $k^n : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a bounded, Borel measurable function. Let $\tau_i^n, i \in \mathbb{N}$, be the time when the i -th customer starts its service. Then for $t \geq 0$, the remaining service of i -th customer is given $v_i^n - \int_{\tau_i^n}^t k^n(\bar{X}_s^n) ds$ (non positive quantity implies that customer's job is completed) provided $\tau_i^n \leq t$. We use negative indices to denote the customers in system at time $t = 0$. Let X_0^n denote the number customers at time $t = 0$ with remaining job $\tilde{v}_i^n, i = -X_0^n + 1, \dots, -[X_0^n - n]^+$, for the customers in service at time $t = 0$ where $\tilde{v}_i^n, i = -X_0^n + 1, \dots, -[X_0^n - n]^+$, are some random variables defined on $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$. Also let Q_0^n denote the number of customers in the queue at time $t = 0$. Hence $Q_0^n = [X_0^n - n]^+$.

For $t \geq 0$, \mathcal{Z}_t^n denotes a measure in \mathcal{M}_+ such that $\mathcal{Z}_t^n(C)$ denotes the number of customers in service with remaining service requirement in C for $C \subset ((0, \infty))$. Hence the total number of customers in service at time t is given by $Z_t^n = \mathcal{Z}_t^n(0, \infty)$. Let Q_t^n be the number of customers waiting in the queue. Define $B_t^n = E_t^n - Q_t^n$. Then $B_t^n + 1$ denotes the index of the head of the customers in the queue waiting to be served. For $0 \leq s \leq t \leq T$, we define $S^n(s, t) = \int_s^t k^n(\bar{X}_u^n) du$. Hence a precise description of \mathcal{Z}^n is given by

$$\mathcal{Z}_t^n(C) = \sum_{i=-X_0^n+1}^{-Q_0^n} \delta_{\tilde{v}_i^n}(C + S^n(0, t)) + \sum_{i=-Q_0^n+1}^{B_t^n} \delta_{v_i^n}(C + S^n(\tau_i^n, t)), \quad (1)$$

for $C \in \mathcal{B}((0, \infty))$. Additional obvious relation satisfied by the processes, for $t \geq 0$, are as follows:

$$X_t^n = Q_t^n + Z_t^n, \quad (2)$$

$$Q_t^n = [X_t^n - n]^+. \quad (3)$$

It is easy to see from (2) and (3) that $Z_t^n = X_t^n \wedge n$. We extend \mathcal{Z}_t^n on $\mathcal{B}([0, \infty))$ by setting $\mathcal{Z}_t^n(\{0\}) = 0$. It is easy to see that \mathcal{Z}^n takes values in $D([0, \infty), \mathcal{M})$.

2.1 The fluid model

In this section, we define the fluid model and state our main theorem. We also state the set of assumptions that are used to prove this result. We define

$$\bar{X}_t^n = \frac{X_t^n}{n}, \bar{Z}_t^n = \frac{1}{n} Z_t^n, \bar{Q}_t^n = \frac{Q_t^n}{n}, \bar{Z}_t^n = \frac{Z_t^n}{n}.$$

The fluid scaling of arrival process E_t^n is define as $\bar{E}_t^n = \frac{1}{n}E_t^n$. Thus the fluid pre-limit equations are (in analogy with (1)–(3)) given by

$$\bar{Z}_t^n(C) = \frac{1}{n} \sum_{i=-X_0^n+1}^{-Q_0^n} \delta_{\bar{v}_i^n}(C + S^n(0, t)) + \frac{1}{n} \sum_{i=-Q_0^n+1}^{B_t^n} \delta_{v_i^n}(C + S^n(\tau_i^n, t)), \quad (4)$$

for $C \in \mathcal{B}([0, \infty))$.

$$\bar{X}_t^n = \bar{Q}_t^n + \bar{Z}_t^n, \quad (5)$$

$$\bar{Q}_t^n = [\bar{X}_t^n - 1]^+. \quad (6)$$

We assume the following conditions:

Condition 2.1 (a) $\frac{\lambda^n}{n} \rightarrow \lambda$ for some $\lambda \in [0, \infty)$ and $\bar{E}_t^n \Rightarrow \lambda t$ in the sense of distribution in $D([0, \infty), \mathbb{R}_+)$;

(b) There exists a probability measure ν with bounded, Lipschitz continuous, density $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\nu^n \rightarrow \nu$ as $n \rightarrow \infty$;

(c) $(\bar{Z}_0^n, \bar{Q}_0^n) \Rightarrow (Z_0, Q_0)$ in $\mathcal{M} \times \mathbb{R}_+$ as $n \rightarrow \infty$ where the function $F(x) = Z_0([x, \infty))$ is Lipschitz continuous on \mathbb{R}_+ and (Z_0, Q_0) is a deterministic element in $\mathcal{M} \times \mathbb{R}_+$.

Since λt is a continuous, deterministic path, one obtains the convergence of the scaled arrival process in probability i.e., for any $T, \epsilon > 0$

$$\lim_{n \rightarrow \infty} \mathbb{P}_n(\sup_{0 \leq t \leq T} |\bar{E}_t^n - \lambda t| > \epsilon) = 0. \quad (7)$$

We impose the following condition on the state dependent service rates:

Condition 2.2 There exists a bounded, Lipschitz continuous map $k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $k^n \rightarrow k$ as $n \rightarrow \infty$ uniformly on compact subsets of \mathbb{R}_+ .

Let $G(\cdot)$ be the distribution function of ν i.e., $G(x) = \nu([0, x])$. Define $G^c(x) = 1 - G(x)$. Also from (5), (6) and Condition 2.1(c) the followings hold:

$$\langle 1, Z_0 \rangle = Z_0, \quad X_0 = Q_0 + Z_0, \quad Q_0 = [X_0 - 1]^+.$$

Theorem 2.1 Assume Conditions 2.1, 2.2 to hold. Then as $n \rightarrow \infty$, $(\bar{Z}^n, \bar{Q}^n) \Rightarrow (Z, Q)$ in $D([0, \infty), \mathcal{M} \times \mathbb{R}_+)$ where (Z, Q) is uniquely defined by the followings:

- for all $t \geq 0$,

$$Z_t((0, \infty)) = Z_t, \quad X_t = Q_t + Z_t, \quad Q_t = [X_t - 1]^+, \quad (8)$$

- for all $t \geq 0$ and $x \geq 0$,

$$Z_t(\bar{C}_x) = F(x + S(0, t)) + \int_0^t G^c(x + S(s, t)) dB_s, \quad (9)$$

where $S(s, t) = \int_s^t k(X_u) du$ and $B_t = \lambda t - Q_t$. We refer (9) as fluid model equation and will be denoted by (k, λ, ν) .

Proof: The existence of a limit satisfying the above properties will be proved in Section 3 and 4. So it is enough to prove the uniqueness of the limit here. Since B_t^n is nondecreasing and we can have pointwise convergence for each subsequential limit (by Skorohod representation theorem), B_t is also non-decreasing. Therefore (9) makes sense. Also for any $a > 0$,

$$\mathcal{Z}_t([0, a)) \leq \mathcal{O}(a),$$

implying $\mathcal{Z}_t(\{0\}) = 0$ for all $t \geq 0$. Now putting $x = 0$ in (9), we have¹

$$\begin{aligned} \mathcal{Z}_t(C_0) &= F(S(0, t)) + \int_0^t G^c(S(s, t)) dB_s \\ &= F(S(0, t)) + \lambda \int_0^t G^c(S(s, t)) ds - \int_0^t G^c(S(s, t)) dQ_s \\ &= F(S(0, t)) + \lambda \int_0^t G^c(S(s, t)) ds - Q_t G^c(0) + Q_0 G^c(S(0, t)) - \int_0^t g(S(s, t)) Q_s d(S(s, t)), \end{aligned}$$

where we used integration-by-parts formula in the last line. Since $G^c(0) = 1$, using (8) we have

$$\begin{aligned} X_t = Z_t + Q_t &= F(S(0, t)) + Q_0 G^c(S(0, t)) + \lambda \int_0^t G^c(S(s, t)) ds \\ &\quad - \int_0^t (X_s - 1)^+ g(S(s, t)) d(S(s, t)). \end{aligned}$$

By our assumptions on $F(\cdot)$, $G(\cdot)$ and $k(\cdot)$, we see that $S(\cdot, t)$ is Lipschitz continuous on $[0, t]$ and hence X_t is continuous in t . Therefore

$$\begin{aligned} X_t = Z_t + Q_t &= F(S(0, t)) + Q_0 G^c(S(0, t)) + \lambda \int_0^t G^c(S(s, t)) ds \\ &\quad + \int_0^t (X_s - 1)^+ k(X_s) g(S(s, t)) ds. \end{aligned}$$

Hence by (8), Z_t, Q_t are continuous in t . Now defining $H_1(x) = F(x) + Q_0 G^c(x)$, $H_2(x) = \lambda G^c(x)$, $H_4(x) = g(x)$, $H_5(x) = k(x)$ for $x \geq 0$ and extending these maps on $(-\infty, 0]$ by their respective values at 0, we have

$$X_t = H_1(S(0, t)) + \int_0^t H_2(S(s, t)) ds + \int_0^t (X_s - 1)^+ H_4(X_s) H_5(S(s, t)) ds. \quad (10)$$

By Lemma 5.9 in Appendix A, X_t is uniquely defined on $[0, T]$ for all $T > 0$. Since $Q_t = [X_t - 1]^+$, Q_t and hence B_t is unique. Therefore (9) implies that \mathcal{Z}_t is uniquely defined on \bar{C}_x . Since $\{\bar{C}_x, x \in \mathbb{R}_+\}$ defines uniquely a Borel measure on $(\mathbb{R}_+, \mathcal{B}([0, \infty)))$, \mathcal{Z}_t is uniquely defined by (9). Hence (\mathcal{Z}, Q) is unique in $D([0, \infty), \mathcal{M} \times \mathbb{R}_+)$. \square

Remark 2.1 One can relax the conditions on $G(\cdot)$ depending on the properties of service rate $k(\cdot)$. For example, if $k(\cdot)$ is constant then it is enough to impose continuity on $G(\cdot)$ instead of Condition 2.1(b) (see [18, 17]).

¹corrected with - below, last line

3 Tightness of the pre-limit processes

In this section, we study the compactness properties of the pre-limit processes. From (4), we get the following equation

$$\bar{\mathcal{Z}}_t^n(C) = \bar{\mathcal{Z}}_s^n(C + S^n(s, t)) + \frac{1}{n} \sum_{i=B_s^n+1}^{B_t^n} \delta_{v_i^n}(C + S^n(\tau_i^n, t)), \quad (11)$$

for all $0 \leq s \leq t$ and $C \in \mathcal{B}([0, \infty))$. Define $\bar{E}^n(s, t) = \bar{E}_t^n - \bar{E}_s^n$. From (7), it is easy to see that given $T, \epsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}_n \left(\sup_{0 \leq s \leq t \leq T} |\bar{E}^n(s, t) - \lambda(t - s)| \leq \epsilon \right) = 1. \quad (12)$$

Defining $\bar{B}_t^n = \frac{B_t^n}{n}$, we have

$$\bar{B}_t^n = \bar{E}_t^n - \bar{Q}_t^n. \quad (13)$$

We recall the following characterization of compact subsets of \mathcal{M} in Prohorov topology from [3] (Theorem A2.4.I).

Definition 3.1 *A set $\mathbf{K} \subset \mathcal{M}$ is relatively compact if and only if $\sup_{\mu \in \mathbf{K}} \mu(\mathbb{R}_+) < \infty$ and there exists a sequence of nested compact sets $C_j \subset \mathbb{R}_+$ such that $\cup C_j = \mathbb{R}_+$ and*

$$\lim_{j \rightarrow \infty} \sup_{\mu \in \mathbf{K}} \mu(C_j^c) = 0.$$

The proof of the following lemma is same as [18, Lemma 5.1].

Lemma 3.1 *Fix $T > 0$. There exists a sequence $\{\epsilon_E(n)\}$ such that $\epsilon_E(n) \rightarrow 0$ as $n \rightarrow \infty$ and*

$$\mathbb{P}_n \left(\sup_{0 \leq s \leq t \leq T} |\bar{E}^n(s, t) - \lambda(t - s)| \leq \epsilon_E(n) \right) \geq 1 - \epsilon_E(n),$$

for all $n \in \mathbb{N}$.

We define $\Omega_E^n = \{\sup_{0 \leq s \leq t \leq T} |\bar{E}^n(s, t) - \lambda(t - s)| \leq \epsilon_E(n)\}$. The following lemma proves compact containment properties of $(\bar{\mathcal{Z}}^n, \bar{Q}^n)$.

Lemma 3.2 *Assume Condition 2.1 to hold. Fix $T > 0$. Then for any positive η there exists a compact set $\mathbf{K} \subset \mathcal{M}$ and $K > 0$ such that*

$$\liminf_{n \rightarrow \infty} \mathbb{P}_n(\bar{\mathcal{Z}}_t^n \in \mathbf{K} \text{ and } \bar{Q}_t^n \leq K \text{ for all } t \in [0, T]) \geq 1 - \eta.$$

Proof: By Condition 2.1(c), there exists a positive integer M_0 such that

$$\sup_n \mathbb{P}_n(\bar{Q}_0^n > M_0) < \frac{\eta}{8}.$$

Since $\bar{Q}_t^n \leq \bar{Q}_0^n + \bar{E}_t^n$, choosing $K = M_0 + 2\lambda T$ we get from (12) that

$$\limsup_{n \rightarrow \infty} \mathbb{P}_n(\bar{Q}_t^n > K \text{ for all } t \in [0, T]) < \frac{\eta}{4}. \quad (14)$$

Define $\vartheta_t^n = \frac{1}{n} \sum_{i=-Q_0^n+1}^{E_t^n} \delta_{v_i^n} \in \mathcal{M}$. From (11), we note that

$$\bar{\mathcal{Z}}_t^n(C_x) \leq \bar{\mathcal{Z}}_0^n(C_x) + \vartheta_t^n(C_x), \quad (15)$$

for all $x \in \mathbb{R}_+$ and $t \geq 0$. For $m \in \mathbb{Z}, \ell \geq 0$, define $\mathcal{L}^n(m, \ell) = \frac{1}{n} \sum_{i=1+m}^{m+\lfloor n\ell \rfloor} \delta_{v_i^n}$. Recall the definition of $\Omega_A^n(M, L)$ (see (B2) in Appendix) and function \bar{f} from Appendix B. Define $\Omega_1^n = \{\bar{Q}_0^n \leq M_0\}$. Therefore using Lemma 3.1 we have for all large n ,

$$\mathbb{P}_n(\Omega_E^n \cap \Omega_1^n \cap \Omega_A^n(M_0 + 1, K)) \geq 1 - \frac{\eta}{4}.$$

Choose n large enough so that $\epsilon_A(n) \leq 1$. Then on $\Omega_E^n \cap \Omega_1^n \cap \Omega_A^n(M_0 + 1, K)$ we have for all large n

$$\langle \bar{f}, \vartheta_t^n \rangle \leq K \langle \bar{f}, \nu^n \rangle + 1 \leq \langle \bar{f}, \bar{\nu} \rangle + 1 \leq M_1, \quad (16)$$

for some positive constant M_1 . Hence using Markov's inequality and (16) we get

$$\vartheta_t^n(C_x) \leq \frac{1}{\bar{f}(x)} M_1, \quad (17)$$

on $\Omega_E^n \cap \Omega_1^n \cap \Omega_A^n(M_0 + 1, K)$ for all $t \in [0, T]$ and all n large. Again by Condition 2.1(c), there exists a compact $\mathbf{K}_0 \subset \mathcal{M}$ such that for all large n

$$\mathbb{P}_n(\bar{\mathcal{Z}}_0^n \in \mathbf{K}_0) \geq 1 - \frac{\eta}{4}. \quad (18)$$

We denote the above event by Ω_2^n . By Definition 3.1, there exists a bounded function $\varrho : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\lim_{x \rightarrow \infty} \varrho(x) = 0$ and on Ω_2^n ,

$$\bar{\mathcal{Z}}_0^n(\mathbb{R}_+) \leq \rho(0), \quad \bar{\mathcal{Z}}_0^n(C_x) \leq \varrho(x) \quad \text{for all } x \in \mathbb{R}_+. \quad (19)$$

We define

$$\mathbf{K} = \{\mu \in \mathcal{M} : \mu(\mathbb{R}_+) \leq \rho(0) + K, \mu(C_x) \leq \varrho(x) + \frac{1}{\bar{f}(x)} M_1 \ \forall x \in \mathbb{R}_+\}.$$

By the property of ϱ and \bar{f} , \mathbf{K} is a compact subset of \mathcal{M} . From (15), (17) and (19) we see that for all large n , on $\Omega_E^n \cap \Omega_1^n \cap \Omega_A^n(M_0 + 1, K) \cap \Omega_2^n$,

$$\bar{\mathcal{Z}}_t^n \in \mathbf{K},$$

for all $t \in [0, T]$. Also for all large n , $\mathbb{P}_n(\Omega_E^n \cap \Omega_1^n \cap \Omega_A^n(M_0 + 1, K) \cap \Omega_2^n) \geq 1 - \frac{\eta}{2}$. Hence the proof follows from (14). \square

Lemma 3.3 *Assume Conditions 2.1, 2.2 to hold. Fix $T > 0$. Then for each $\epsilon, \eta > 0$, there exists a $\kappa > 0$ such that*

$$\liminf_{n \rightarrow \infty} \mathbb{P}_n(\sup_{[0, T]} \sup_{x \in \mathbb{R}_+} \bar{\mathcal{Z}}_t^n([x, x + \kappa]) \leq \epsilon) > 1 - \eta. \quad (20)$$

Proof: Using Condition 2.1(c), one can prove that for any $\epsilon, \eta > 0$ there exists a positive κ such that

$$\liminf_{n \rightarrow \infty} \mathbb{P}_n \left(\sup_{x \in \mathbb{R}_+} \bar{\mathcal{Z}}_0^n([x, x + \kappa]) \leq \epsilon/2 \right) \geq 1 - \frac{\eta}{4}. \quad (21)$$

In fact, the proof is same as the proof of (79) in [18]. We denote the above event by Ω_3^n and the event in Lemma 3.2 by Ω_4^n . Define $\Omega_5^n = \Omega_3^n \cap \Omega_4^n \cap \Omega_E^n \cap \Omega_A^n(M, L)$ for $L = 2\lambda T + K$ and $M = \lfloor L \rfloor + 1$. From Lemma 3.2, it is easy to see that there exists $K > 0$ such that

$$\liminf_{n \rightarrow \infty} \mathbb{P}_n(\Omega_5^n) \geq 1 - \eta.$$

From (11), we see that for any $\kappa > 0$ and $t \in [0, T]$

$$\bar{\mathcal{Z}}_t^n([x, x + \kappa]) = \bar{\mathcal{Z}}_0^n([x, x + \kappa] + S^n(0, t)) + \frac{1}{n} \sum_{i=-Q_0^n+1}^{B_t^n} \delta_{v_i^n}([x, x + \kappa] + S^n(\tau_i^n, t)). \quad (22)$$

On Ω_3^n , we have $\sup_{x \in \mathbb{R}_+} \bar{\mathcal{Z}}_0^n([x, x + \kappa]) \leq \epsilon/2$. Hence choosing $x = x(\omega) = x + S^n(0, t)$ we have on Ω_3^n , $\sup_{x \in \mathbb{R}_+} \bar{\mathcal{Z}}_0^n([x, x + \kappa] + S^n(0, t)) \leq \epsilon/2$. Thus we need to estimate the second term on Ω_5^n . So we denote the second term by Ξ_t .

For any $\delta > 0$, we consider a partition $0 = t_0 < t_1 < \dots < t_r = t$ of $[0, t]$ with $|t_{i+1} - t_i| < \delta$ for $k = 0, 1, 2, \dots, r-1$. Since $-Q_0^n = B_0^n$, we have

$$\Xi_t = \sum_{k=0}^{r-1} \frac{1}{n} \sum_{i=B_{t_k}^n+1}^{B_{t_{k+1}}^n} \delta_{v_i^n}([x, x + \kappa] + S^n(\tau_i^n, t)).$$

From (B2), on $\Omega_A^n(M, L)$, we have for all $k = 0, 1, 2, \dots, r-1$,

$$-Q_0^n \leq B_{t_i}^n \leq E_t^n, \text{ and so, } 0 \leq B_{t_{i+1}}^n - B_{t_i}^n \leq E_t^n + Q_0^n,$$

and

$$\max_{-nM < m < nM} \sup_{\ell \in [0, L]} \sup_{f \in \mathcal{V}} |\langle f, \mathcal{L}^n(m, \ell) \rangle - \ell \langle f, \nu^n \rangle| \leq \epsilon_A(n).$$

Hence for all $m \in (-nM, nM)$, $\ell \in [0, L]$ and for all $a, b \in \mathbb{R}_+$, $a \leq b$, we have

$$\langle \chi_{[a, b]}, \mathcal{L}(m, \ell) \rangle \leq \ell \langle \chi_{[a, b]}, \nu^n \rangle + 2\epsilon_A(n), \quad (23)$$

on $\Omega_A^n(M, L)$. Now for $t_k \leq \tau_i^n \leq t_{k+1}$, $[x, x + \kappa] + S^n(\tau_i^n, t) \subset [x + S^n(t_{k+1}, t), x + \kappa + S^n(t_k, t)]$. Now fixing $a = x + S^n(t_{k+1}, t)$, $b = x + \kappa + S^n(t_k, t)$ and observing that, on $\Omega_4^n \cap \Omega_E^n$, $B_{t_i} \in (-nM, nM)$ (for above choice of M) and $\bar{B}_{t_{i+1}}^n - \bar{B}_{t_i}^n \in [0, L]$ for all n large, we have

$$\frac{1}{n} \sum_{i=B_{t_k}^n+1}^{B_{t_{k+1}}^n} \delta_{v_i^n}([x, x + \kappa] + S^n(\tau_i^n, t)) \leq (\bar{B}_{t_{i+1}}^n - \bar{B}_{t_i}^n) \nu^n([x + S^n(t_{k+1}, t), x + \kappa + S^n(t_k, t)]) + 2\epsilon_A(n),$$

for $k = 0, 1, 2, \dots, r-1$. Since $\nu^n \rightarrow \nu$ in Prohorov metric, for any $\epsilon_1 > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, and closed set $C \subset \mathbb{R}_+$ (see Notations)

$$\nu^n(C) \leq \nu(C^{\epsilon_1}) + \epsilon_1.$$

Hence combining above two we have

$$\begin{aligned} \frac{1}{n} \sum_{i=B_{t_i}^n+1}^{B_{t_{i+1}}^n} \delta_{v_i^n}([x, x+\kappa] + S^n(\tau_i^n, t)) &\leq (\bar{B}_{t_{i+1}}^n - \bar{B}_{t_i}^n) \nu([x + S^n(t_{k+1}, t) - \epsilon_1, x + \kappa + S^n(t_k, t) + \epsilon_1]) \\ &\quad + (\bar{B}_{t_{i+1}}^n - \bar{B}_{t_i}^n) \epsilon_1 + 2\epsilon_A(n), \end{aligned}$$

on $\Omega_4^n \cap \Omega_E^n$ for all n large. At this point, we observe that $\sup_{s \in [0, T]} \bar{X}_s^n \leq \sup_{s \in [0, T]} \bar{Q}_s^n + 1 \leq L+1$ on Ω_4^n for all large n . Hence by Condition 2.2, on Ω_4^n , $\sup_{[0, T]} k^n(\bar{X}_s^n) \leq \sup_{[0, L+1]} k^n(x) \leq \sup_{[0, L+1]} k(x) + 1 < M_2$ for some positive constant M_2 and for all n large. Hence $|S^n(t_k, t) - S^n(t_{k+1}, t)| \leq \delta M_2$ on Ω_4^n for all n large. By Condition 2.1(b), we can choose δ and κ small enough so that on Ω_4^n

$$\nu([x + S^n(t_{k+1}, t) - \epsilon_1, x + \kappa + S^n(t_k, t) + \epsilon_1]) < \frac{\epsilon}{8L},$$

for all ϵ_1 small enough and all n large. Hence summing up the above expression we have for all $t \in [0, T]$

$$\Xi_t \leq L \cdot \frac{\epsilon}{8L} + L\epsilon_1 + 2r\epsilon_A(n),$$

on Ω_5^n for all n large. Since $\epsilon_1, \epsilon_A(n)$ do not depend on r and x , we can choose them small to make the right hand side smaller than $\frac{\epsilon}{2}$ for all n large and $x \in \mathbb{R}_+, t \in [0, T]$. The proof is done from (22) and definition of Ω_5^n . \square

For any path $\phi \in D([0, \infty), E)$ where (E, π) is polish space, the δ -oscillation of ϕ on $[0, T]$, $T > 0$, is defined as follows

$$w(\phi, \delta, T) = \sup_{s, t \in [0, T], |s-t| \leq \delta} \pi(\phi(s), \phi(t)).$$

The following lemma gives the oscillation bounds on the stochastic process

Lemma 3.4 *Assume Conditions 2.1, 2.2 to hold. Fix $T > 0$. Then for each $\epsilon, \eta > 0$, there exists $\delta > 0$ such that*

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mathbb{P}_n(w(\bar{Z}^n, \delta, T) \leq \epsilon) &\geq 1 - \eta, \\ \liminf_{n \rightarrow \infty} \mathbb{P}_n(w(\bar{Q}^n, \delta, T) \leq \epsilon) &\geq 1 - \eta. \end{aligned}$$

Proof: Let $t, s \in [0, T]$, $s \leq t$ and $|t-s| \leq \delta$. Let D_t^n be the number of customers finished their job by time t . Then it is easy to see that

$$D_t^n - D_s^n \leq \mathcal{Z}_s^n[0, S^n(s, t)] + \sum_{i=B_s^n+1}^{B_t^n} \delta_{v_i^n}([0, S^n(s, t)]).$$

Recall the event Ω_5^n from Lemma 3.3. By definition, on Ω_5^n , we have $B_s^n \in (-nM, nM)$ and $\bar{B}_t^n - \bar{B}_s^n \in [0, L]$ for all n large. Since $\sup_{[0, T]} \bar{X}_s^n \leq \sup_{[0, T]} \bar{Q}_s^n + 1 \leq L+1$, we can choose

δ small enough so that $|S^n(s, t)| \leq \kappa_1$ on Ω_5^n for all n large where $\kappa_1 = \kappa_1(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Therefore by the definition of $\Omega_A^n(M, L)$ and (23) we have

$$\frac{1}{n} \sum_{i=B_s^n+1}^{B_t^n} \delta_{v_i^n}([0, S^n(s, t)]) \leq L \langle \chi_{[0, \kappa_1]}, \nu^n \rangle + 2\epsilon_A(n),$$

on Ω_5^n for all n large. Since $\nu^n \rightarrow \nu$, by the same reasoning as in Lemma 3.3, we can choose δ small so that

$$\frac{1}{n} \sum_{i=B_s^n+1}^{B_t^n} \delta_{v_i^n}([0, S^n(s, t)]) \leq \epsilon,$$

on Ω_5^n for all n large. If we denote the event in (20) by Ω_6^n , then on $\Omega_6^n \cap \Omega_5^n$

$$\bar{Z}_s^n[0, S^n(s, t)] \leq \bar{Z}_s^n[0, \kappa_1] \leq \epsilon$$

for all n large and δ chosen small enough. Hence with this choice of δ , $\mathbb{P}_n(\Omega_6^n \cap \Omega_5^n) \geq 1 - 2\eta$ and

$$\frac{1}{n}(D_t^n - D_s^n) \leq 2\epsilon$$

for all n large. Since $X_t^n = X_0^n + E_t^n - D_t^n$, we have $|\bar{X}_s^n - \bar{X}_t^n| \leq |\bar{E}_s^n - \bar{E}_t^n| + \frac{1}{n}(D_t^n - D_s^n) \leq 3\epsilon$ on $\Omega_6^n \cap \Omega_5^n$ for all n large provided δ chosen small enough. Therefore with this choice of δ , we have (using (6))

$$|\bar{Q}_s^n - \bar{Q}_t^n| \leq 3\epsilon$$

on $\Omega_6^n \cap \Omega_5^n$ for all n large. Hence the second claim follows by replacing ϵ, η with $\epsilon/3, \eta/2$ respectively.

Now we prove the first claim. We note that for any $t, s \in [0, T]$, $|\bar{B}_s^n - \bar{B}_t^n| \leq |\bar{E}_s^n - \bar{E}_t^n| + |\bar{Q}_s^n - \bar{Q}_t^n|$ (from (13)). Hence on $\Omega_6^n \cap \Omega_5^n$, $|\bar{B}_s^n - \bar{B}_t^n| \leq 4\epsilon$ for all n large provided $|t - s| \leq \delta \wedge \frac{\epsilon}{2\lambda}$. Let $C \subset \mathbb{R}^+$ be closed and C^ϵ be its ϵ -enlargement. Now choose δ small enough so that $\kappa_1 < \epsilon$ and so $C + S^n(s, t) \subset C^{\kappa_1} \subset C^\epsilon$ on Ω_5^n for all n large. Hence from (11) we have

$$\bar{Z}_t^n(C) - \bar{Z}_s^n(C^\epsilon) \leq \bar{Z}_s^n(C + S^n(s, t)) - \bar{Z}_s^n(C^\epsilon) + |\bar{B}_s^n - \bar{B}_t^n| \leq 4\epsilon,$$

on $\Omega_6^n \cap \Omega_5^n$ for all n large. Again for any $c \in C$ and $S^n(s, t) \leq \kappa_1$, we have $\text{dist}(c - S^n(s, t), C) < \epsilon$ implying $c \in C^\epsilon + S^n(s, t)$ and so $C \subset C^\epsilon + S^n(s, t)$. Hence from (11)

$$\bar{Z}_s^n(C) - \bar{Z}_t^n(C^\epsilon) \leq \bar{Z}_s^n(C) - \bar{Z}_s^n(C^\epsilon + S^n(s, t)) \leq 0,$$

on $\Omega_6^n \cap \Omega_5^n$ for all n large. Hence for all closed set $C \in \mathcal{B}([0, \infty))$ we have

$$\bar{Z}_s^n(C) \leq \bar{Z}_t^n(C^{4\epsilon}) + 4\epsilon \text{ and } \bar{Z}_t^n(C) \leq \bar{Z}_s^n(C^{4\epsilon}) + 4\epsilon,$$

on $\Omega_6^n \cap \Omega_5^n$ for all n large. Hence $\rho(\bar{Z}_s^n, \bar{Z}_t^n) \leq 4\epsilon$ on $\Omega_6^n \cap \Omega_5^n$ for all n large. Thus first claim follows by replacing ϵ, η with $\epsilon/4, \eta/2$ respectively. \square

Now we introduce a weaker oscillation function w' on $D([0, \infty), \mathcal{M} \times \mathbb{R}_+)$. Define metric $d'(\cdot, \cdot) = \max\{\rho(\cdot, \cdot), |\cdot| \}$ on $\mathcal{M} \times \mathbb{R}_+$ which induces a separable complete metric on it. For any $\psi \in D([0, \infty), \mathcal{M} \times \mathbb{R}_+)$ and $T, \delta > 0$ define

$$w'(\psi, \delta, T) = \inf_{t_i} \max_i \sup_{s, t \in [t_{i-1}, t_i]} d'(\psi(s), \psi(t)),$$

where $\{t_i\}$ ranges over all partition of the form $0 = t_0 < t_1 < \dots < t_j = T$ with $\min_{1 \leq i \leq j} (t_i - t_{i-1}) > \delta$ and $j \geq 1$. It is easy to see that for $\delta > 0$ we can have a partition t_i of $[0, T]$ such that $\min_{1 \leq i \leq j} (t_i - t_{i-1}) > \delta$ and $\max_{1 \leq i \leq j} (t_i - t_{i-1}) \leq 2\delta$ and hence for any $\psi \in D([0, \infty), \mathcal{M} \times \mathbb{R}_+)$ we have

$$w'(\psi, \delta, T) \leq w(\psi, 2\delta, T). \quad (24)$$

Hence from Lemma 3.4 and (24), we get that for any $T, \epsilon, \eta > 0$, there exists $\delta > 0$ such that

$$\liminf_{n \rightarrow \infty} \mathbb{P}_n(w'((\bar{\mathcal{Z}}^n, \bar{Q}^n), \delta, T) \leq \epsilon) \geq 1 - \eta. \quad (25)$$

For $\psi \in D([0, \infty), \mathcal{M} \times \mathbb{R}_+)$, define

$$J(\psi) = \int_0^\infty e^{-s} [J(\psi, s) \wedge 1] ds, \text{ where } J(\psi, t) = \sup_{0 \leq s \leq t} d'(\psi(s), \psi(s-)).$$

Again it is easy to see that $J(\psi, t) \leq w(\psi, \delta, t)$ for any $\delta > 0$ and hence $J(\psi) \leq w(\psi, \delta, T) + e^{-T}$ for all $T, \delta > 0$. Thus applying Lemma 3.4, we get that for any $\epsilon, \eta > 0$,

$$\liminf_{n \rightarrow \infty} \mathbb{P}_n(J((\bar{\mathcal{Z}}^n, \bar{Q}^n)) \leq \epsilon) \geq 1 - \eta. \quad (26)$$

Now we note that the process $(\bar{\mathcal{Z}}^n, \bar{Q}^n)$ satisfies the conditions (a) compact containment property (Lemma 3.2) and (b) oscillation bound ((25)) of Corollary 3.7.4 in [5]. Hence $(\bar{\mathcal{Z}}^n, \bar{Q}^n)$ is tight in $D([0, \infty), \mathcal{M} \times \mathbb{R}_+)$ and $(\bar{\mathcal{Z}}^n, \bar{Q}^n) \Rightarrow (\bar{\mathcal{Z}}, \bar{Q})$ along some subsequence for some random variable $(\bar{\mathcal{Z}}, \bar{Q})$ taking values in $D([0, \infty), \mathcal{M} \times \mathbb{R}_+)$. Also (26) satisfies the condition (a) in Theorem 3.10.2 in [5] which implies that (\mathcal{Z}, Q) has continuous paths almost surely.

4 Characterization of the limits

In this section, we characterize some properties of the limits which lead to uniqueness. To have simple notations, we consider full sequence to converge instead of subsequence. To this end, we intend to define all the variable on a common probability space using Skorohod representation theorem. From Condition 2.1(a), it is clear that \bar{E}^n is tight in $D([0, \infty), \mathbb{R}_+)$. Hence $(\bar{\mathcal{Z}}^n, \bar{Q}^n, \bar{E}^n)$ is tight in $D([0, \infty), \mathcal{M} \times \mathbb{R}_+) \times D([0, \infty), \mathbb{R}_+)$. Therefore by Skorohod representation theorem we can say that $(\bar{\mathcal{Z}}^{1n}, \bar{Q}^{1n}, \bar{E}^{1n}, U^{1n}, V^{1n}) \rightarrow (\mathcal{Z}^1, Q^1, \lambda)$ in $D([0, T], \mathcal{M} \times \mathbb{R}_+) \times D([0, \infty), \mathbb{R}_+)$ almost surely on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ where

$$\text{law of } (\bar{\mathcal{Z}}^{1n}, \bar{Q}^{1n}, \bar{E}^{1n}) = \text{law of } (\bar{\mathcal{Z}}^n, \bar{Q}^n, \bar{E}^n) \text{ for all } n,$$

and law of $(\mathcal{Z}^1, Q^1) = \text{law of } (\mathcal{Z}, Q)$.

Also (\mathcal{Z}^1, Q^1) has continuous paths almost surely for $Z_t^1 = \langle 1, \mathcal{Z}_t^1 \rangle$. Hence for any $T > 0$, the followings hold, almost surely:

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, T]} \rho(\mathcal{Z}_s^{1n}, \mathcal{Z}_s^1) = 0 \quad (27)$$

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, T]} |\bar{Q}_s^{1n} - Q_s^1| = 0,$$

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, T]} |\bar{\mathcal{Z}}_s^{1n} - \mathcal{Z}_s^1| = 0,$$

$$\lim_{n \rightarrow \infty} \sup_{s \in [0, T]} |\bar{B}_s^{1n} - B_s^1| = 0, \quad (28)$$

where $Z_t^1 = \langle 1, \mathcal{Z}_t^1 \rangle$, $B_t^{1n} = E_1^{1n} - Q_t^{1n}$ and $B_t^1 = \lambda t - Q_t^1$. Hence from (5) and (6), we get $\bar{X}^{1n} \rightarrow X^1$ uniformly on $[0, T]$, $T > 0$, and

$$X_t^1 = Q_t^1 + Z_t^1 \text{ and } Q_t^1 = [X_t^1 - 1]^+.$$

(28) implies that B_t^1 is nondecreasing in t and so it is a function of bounded variation on $[0, T]$ for all $T > 0$. Also from above, it is easy to see that

$$\text{law of } (\mathcal{Z}^1, Q^1, B^1, X^1) = \text{law of } (\mathcal{Z}, Q, B, X).$$

Lemma 4.5 *Let (\mathcal{S}, π) be a metric space and $K \subset \mathcal{S}$ be compact. Let $f : \mathcal{S} \rightarrow \mathbb{R}$ be a function satisfying the following: for any sequence $s_n \rightarrow s$ and $s \in K$, $f(s_n) \rightarrow f(s)$ as $n \rightarrow \infty$. Then for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(s_1) - f(s_2)| < \epsilon$ whenever $s_1 \in \mathcal{S}$, $s_2 \in K$ and $\pi(s_1, s_2) \leq \delta$.*

Proof: If not, then there exists two sequences $\{s_n\}, \{\tilde{s}_n\}$ such that $\{\tilde{s}_n\} \subset K$ and

$$\pi(s_n, \tilde{s}_n) \leq \frac{1}{n}, \quad |f(s_n) - f(\tilde{s}_n)| \geq \epsilon \quad \forall n \geq 1.$$

Now K being compact, there exists $s \in K$ such that along some subsequence $\{n_k\}$, $\tilde{s}_{n_k} \rightarrow s \in K$. Hence $s_{n_k} \rightarrow s \in K$ as $n_k \rightarrow \infty$. This is contradicting to the fact that $|f(s_{n_k}) - f(\tilde{s}_{n_k})| \geq \epsilon$ for all n_k . Hence the proof. \square

Following lemma is a consequence of Lemma 3.3.

Lemma 4.6 *Fix $T > 0$ and $x_0 \in \mathbb{R}_+$. Consider a decreasing sequence $\{f^n\}$ in $C_b(\mathbb{R})$ so that $f^n \geq 0$, $f^n(x) = 1$ on $[x_0 - \frac{1}{n}, x_0 + \frac{1}{n}]$ and f^n vanishes outside of $[x_0 - \frac{2}{n}, x_0 + \frac{2}{n}]$. Then $\tilde{\mathbb{P}}(\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \langle f^n, \mathcal{Z}_t^1 \rangle > 0) = 0$. In particular, for $t \geq 0$, \mathcal{Z}_t^1 has no atom at x_0 almost surely.*

Proof: Let $\tilde{\mathbb{P}}(\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \langle f^n, \mathcal{Z}_t^1 \rangle > \kappa_2) \geq \eta$ for some positive constant κ_2, η . Then $\tilde{\mathbb{P}}(\sup_{t \in [0, T]} \langle f^m, \mathcal{Z}_t^1 \rangle > \kappa_2) \geq \eta$ for all m . Note that $\{\mathcal{Z}_t^1 : t \in [0, T]\}$ is compact in \mathcal{M} . Now from (27) and Lemma 4.5 we get, $\sup_{t \in [0, T]} \langle f^m, \bar{\mathcal{Z}}^{1, n} \rangle \rightarrow \sup_{t \in [0, T]} \langle f^m, \mathcal{Z}_t^1 \rangle$ as $n \rightarrow \infty$ almost surely. Therefor using Fatou's lemma, for all m ,

$$\liminf \mathbb{P}_n(\sup_{t \in [0, T]} \langle f^m, \bar{\mathcal{Z}}_t^n \rangle > \kappa_2) = \liminf \tilde{\mathbb{P}}(\sup_{t \in [0, T]} \langle f^m, \bar{\mathcal{Z}}_t^{1n} \rangle > \kappa_2) \geq \tilde{\mathbb{P}}(\sup_{t \in [0, T]} \langle f^m, \mathcal{Z}_t^1 \rangle > \kappa_2) \geq \eta.$$

Now we choose $\kappa > 0$ from Lemma 3.3 for ϵ, η replaced by $\kappa_2/2, \eta/2$. Thus if we choose m large enough, we get

$$\eta \leq \liminf_{n \rightarrow \infty} \mathbb{P}_n(\sup_{t \in [0, T]} \langle f^m, \bar{\mathcal{Z}}_t^n \rangle > \kappa_2) \leq \liminf_{n \rightarrow \infty} \mathbb{P}_n(\sup_{t \in [0, T]} \bar{\mathcal{Z}}_t^n[(x_0 - \frac{\kappa}{4}) \vee 0, x_0 + \frac{\kappa}{4}] > \kappa_2) \leq \frac{\eta}{2},$$

which is a contradiction. This completes the proof. \square .

An immediate consequence of the above lemma is $\mathcal{Z}_t([0, \infty) = \mathcal{Z}_t((0, \infty)) = Z_t$ for all $t \geq 0$, almost surely.

Lemma 4.7 For any $t \geq 0$, \mathcal{Z}_t satisfies the fluid model equation (k, λ, ν) given by

$$\mathcal{Z}_t(\bar{C}_x) = \mathcal{Z}_0(\bar{C}_x + S(0, t)) + \int_0^t G^c(x + S(s, t)) dB_s,$$

almost surely where $S(s, t) = \int_s^t k(X_u) du$.

Proof: It is enough to prove the above result for the process \mathcal{Z}^1 . For $t \geq 0$ and $C \in \mathcal{B}([0, \infty))$, we have

$$\mathcal{Z}_t^n(C) = \mathcal{Z}_0(C + S^n(0, t)) + I_t^n(C), \quad (29)$$

where

$$I_t^n(C) = \sum_{j=0}^{J-1} \frac{1}{n} \sum_{i=B_{t_j}^n+1}^{B_{t_{j+1}}^n} \delta_{v_i^n}(C + S^n(\tau_i^n, t)),$$

for any partition $\{t_j\}_{j=0}^{j=J(t)}$ of $[0, t]$. But \mathcal{Z}_t^{1n} might not possess the same expression as \mathcal{Z}_t^n as the stochastic variables v_i^n might not make sense on new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Define $S^{1n}(s, t) = \int_s^t k^n(\bar{X}_u^{1n}) du$ and $S^1(s, t) = \int_s^t k(X_u^1) du$. Fix $x \in \mathbb{R}_+$ and let A_T be a countable dense set in $[0, T]$. Then for any $\epsilon > 0$,

$$\begin{aligned} & \tilde{\mathbb{P}}\left(\sup_{t \in A_T} |\mathcal{Z}_t^{1n}(\bar{C}_x) - \mathcal{Z}_0^{1n}(\bar{C}_x + S^{1n}(0, t)) - \int_0^t G^c(x + S^{1n}(s, t)) d\bar{B}_s^{1n}| > \epsilon\right) \\ &= \mathbb{P}_n\left(\sup_{t \in A_T} |\mathcal{Z}_t^n(\bar{C}_x) - \mathcal{Z}_0^n(\bar{C}_x + S^n(0, t)) - \int_0^t G^c(x + S^n(s, t)) d\bar{B}_s^n| > \epsilon\right) \\ &= \mathbb{P}_n\left(\sup_{t \in [0, T]} |I_t^n(\bar{C}_x) - \int_0^t G^c(x + S^n(s, t)) d\bar{B}_s^n| > \epsilon\right). \end{aligned} \quad (30)$$

Applying Lemma 3.2, for any positive η , we have constant K such that $\mathbb{P}_n(\Omega_7^n) \geq 1 - \eta$ where $\Omega_7^n = \{\sup_{t \in [0, T]} \bar{X}_t^n \leq K\}$ for all n large.

We choose $\delta > 0$ and partitions $\{t_i\}_{i=0}^{i=J(t)}$ such that $\max_{1 \leq j \leq J(t)} |t_j - t_{j-1}| < \delta$ and $\sup_{t \in [0, T]} J(t) = J(\delta) < \infty$. Recall that $V^n \rightarrow 0$ as $n \rightarrow \infty$ in probability. From Lemma 3.1 and Lemma 3.2, we choose M such that $\mathbb{P}_n(\Omega_8^n) \geq 1 - \eta$ for all n large where $\Omega_8^n = \{\sup_{[0, T]} \bar{B}_s^n < \frac{M}{2}\}$. Hence on Ω_8^n , $(\bar{B}_{t_{j+1}}^n - \bar{B}_{t_j}^n) < M$ for all n . Again for $i \in \{B_{t_j}^n + 1, B_{t_j}^n + 2, \dots, B_{t_{j+1}}^n\}$, we have $t_j < \tau_i^n \leq t_{j+1}$. Recall the event $\Omega_A^n(M, M)$ from (B2). For any $\epsilon_1 > 0$, we have for $0 \leq j \leq J(t) - 1$,

$$\begin{aligned} & \frac{1}{n} \sum_{i=B_{t_j}^n+1}^{B_{t_{j+1}}^n} \delta_{v_i^n}(\bar{C}_{x+S^n(\tau_i^n, t)}) - \frac{1}{n} \sum_{t_j < \tau_i^n \leq t_{j+1}} G^c(x + S^n(\tau_i^n, t)) \\ & \leq \frac{1}{n} \sum_{i=B_{t_j}^n+1}^{B_{t_{j+1}}^n} \delta_{v_i^n}(\bar{C}_{x+S^n(t_{j+1}, t)}) - (\bar{B}_{t_{j+1}}^n - \bar{B}_{t_j}^n) G^c(x + S^n(t_j, t)) \\ & \leq (\bar{B}_{t_{j+1}}^n - \bar{B}_{t_j}^n) \left(\nu^n(\bar{C}_{x+S^n(t_{j+1}, t)}) - G^c(x + S^n(t_j, t)) \right) + \epsilon_1 \end{aligned}$$

on $\Omega_A^n(M, M) \cap \Omega_8^n$, for all n large. Using the fact that $\rho(\nu^n, \nu) \rightarrow 0$ as $n \rightarrow \infty$, we get on $\Omega_A^n(M, M) \cap \Omega_8^n \cap \Omega_7^n$,

$$\begin{aligned} & \sum_{j=0}^{J-1} \frac{1}{n} \sum_{i=B_{t_j}^n+1}^{B_{t_{j+1}}^n} \delta_{v_i^n}(\bar{C}_x + S^n(\tau_i^n, t)) - \int_0^t G^c(x + S^n(s, t)) d\bar{B}_s^n \\ & \leq \sum_{j=0}^{J-1} (\bar{B}_{t_{j+1}}^n - B_{t_j}^n) \left(\nu(\bar{C}_x + (S(t_{j+1}, t) - \epsilon_1) \vee 0) - G^c(x + S^n(t_j, t)) \right) + M\epsilon_1 + J(t)\epsilon_1, \\ & \leq M|g|_\infty(K\delta + \epsilon_1) + (M + J(\delta))\epsilon_1, \end{aligned}$$

for all n large where $|g|_\infty$ denote the supremum norm of g . First choosing $\delta > 0$ small enough and then choosing ϵ_1 we can have the r.h.s. less than $\epsilon/2$ on $\Omega_A^n(M, M) \cap \Omega_8^n \cap \Omega_7^n$ for all n large and for all $t \in [0, T]$. A similar calculation gives that $I_t^n(\bar{C}_x) - \int_0^t G^c(x + S^n(s, t)) d\bar{B}_s^n \geq -\epsilon/2$ on $\Omega_A^n(M, M) \cap \Omega_8^n \cap \Omega_7^n$ for all n large and for all $t \in [0, T]$. Since $\liminf_{n \rightarrow \infty} \mathbb{P}_n(\Omega_A^n(M, M) \cap \Omega_8^n \cap \Omega_7^n) \geq 1 - 3\eta$, we have from (30)

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{t \in [0, T]} |\mathcal{Z}_t^{1n}(\bar{C}_x) - \mathcal{Z}_0^{1n}(\bar{C}_x + S^{1n}(0, t)) - \int_0^t G^c(x + S^{1n}(s, t)) d\bar{B}_s^{1n}| > \epsilon \right) \leq 3\eta.$$

η begin arbitrary, we have for any $\epsilon > 0$,

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left(\sup_{t \in [0, T]} |\mathcal{Z}_t^{1n}(\bar{C}_x) - \mathcal{Z}_0^{1n}(\bar{C}_x + S^{1n}(0, t)) - \int_0^t G^c(x + S^{1n}(s, t)) d\bar{B}_s^{1n}| > \epsilon \right) = 0. \quad (31)$$

Since $\sup_{[0, T]} |\bar{X}_s^{1n} - X_s^1| \rightarrow 0$, by Condition 2.2, $\sup_{[0, T]} |k^n(\bar{X}_s^{1n}) - k(X_s^1)| \rightarrow 0$ as $n \rightarrow \infty$, almost surely. Hence

$$\sup_{s \in [0, T]} |S^{1n}(s, t) - S^1(s, t)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This implies $\sup_{s \in [0, T]} |G^c(x + S^{1n}(s, t)) - G^c(x + S^1(s, t))| \rightarrow 0$ as $n \rightarrow \infty$. Since $\rho(\mathcal{Z}_0^{1n}, \mathcal{Z}_0^1) \rightarrow 0$ almost surely, we have for $t \in [0, T]$

$$\mathcal{Z}_0^{1n}(\bar{C}_x + S^{1n}(0, t)) \leq \mathcal{Z}_0^1(\bar{C}_x + S^1(0, t) - \epsilon_2) + \epsilon_2, \mathcal{Z}_0^1(\bar{C}_x + S^1(0, t) + \epsilon_2) \leq \mathcal{Z}_0^{1n}(\bar{C}_x + S^{1n}(0, t)) + \epsilon_2,$$

for any chosen $\epsilon_2 > 0$ and all n large (might depend on sample point). By Condition 2.1(c), \mathcal{Z}_0^1 is deterministic with distribution function Lipschitz continuous and so $\sup_{t \in [0, T]} |\mathcal{Z}_0^{1n}(\bar{C}_x + S^{1n}(0, t)) - \mathcal{Z}_0^1(\bar{C}_x + S^1(0, t))| \rightarrow 0$ almost surely. From (28), it is easy to check that $\rho(d\bar{B}^{1n}, dB^1) \rightarrow 0$ where $d\bar{B}^{1n}, dB^1$ are considered as Borel measures on $[0, T]$. Since B^1 is continuous almost surely, applying Theorem A2.3.I in [3] and (28), we get

$$\sup_{t \in [0, T]} \left| \int_0^t G^c(x + S^1(s, t)) d\bar{B}_s^{1n} - \int_0^t G^c(x + S^1(s, t)) dB_s^1 \right| \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ almost surely,}$$

and hence

$$\sup_{t \in [0, T]} \left| \int_0^t G^c(x + S^{1n}(s, t)) d\bar{B}_s^{1n} - \int_0^t G^c(x + S^1(s, t)) dB_s^1 \right| \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ almost surely.}$$

Now we show that $\sup_{t \in [0, T]} |\bar{\mathcal{Z}}_t^{1n}(\bar{C}_x) - \mathcal{Z}_t^1(\bar{C}_x)| \rightarrow 0$ as $n \rightarrow \infty$ almost surely. Consider the map $f : \mathcal{M} \rightarrow \mathbb{R}$ defined by $f(\mu) = \mu(\bar{C}_x)$. From Lemma 4.6, we see that f satisfies the condition of Lemma 4.5 for the compact set $\{\mathcal{Z}_t^1 : t \in [0, T]\}$, almost surely. Hence using (27) and Lemma 4.5, we have $\sup_{t \in [0, T]} |\bar{\mathcal{Z}}_t^{1n}(\bar{C}_x) - \mathcal{Z}_t^1(\bar{C}_x)| \rightarrow 0$ as $n \rightarrow \infty$, almost surely. Combining the above estimates with (31), we get $\Omega(x, T) \in \tilde{\mathcal{F}}$ such that $\tilde{\mathbb{P}}(\Omega(x, T)) = 1$ and

$$\mathcal{Z}_t^1(\bar{C}_x) = \mathcal{Z}_0^1(\bar{C}_x + S^1(0, t)) + \int_0^t G^c(x + S^1(s, t)) dB_s^1,$$

for all $t \in [0, T]$ on $\Omega(x, T)$. Since $\{\bar{C}_x : x \in \mathbb{R}_+, x \text{ rational}\}$ determines any Borel-measure uniquely on \mathbb{R}_+ we can take $\Omega_\infty = \cap_{T \in \mathbb{N}} \cap_{\{x: x \text{ rational}\}} \Omega(x, T)$ on which

$$\mathcal{Z}_t^1(\bar{C}_x) = \mathcal{Z}_0^1(\bar{C}_x + S^1(0, t)) + \int_0^t G^c(x + S^1(s, t)) dB_s^1,$$

for all $t \geq 0$ and $x \in \mathbb{R}_+$. This completes the proof as $\tilde{\mathbb{P}}(\Omega_\infty) = 1$. \square

5 Appendix

5.1 Appendix A

In this section, we prove existence of unique solution to the fluid model type equations.

Lemma 5.8 *Fix $T > 0$. Let $k, H_i : \mathbb{R} \rightarrow \mathbb{R}, i = 1, 2, 3, 4$, be bounded, Lipschitz continuous. Then the following integral equation*

$$\begin{aligned} x_t &= H_1\left(\int_0^t k(x_s) ds\right) + \int_0^t H_2\left(\int_s^t k(x_u) du\right) ds + \int_0^t H_3(x_s) H_4\left(\int_s^t k(x_u) du\right) ds \\ x_0 &= H_1(0). \end{aligned} \quad (32)$$

has a unique solution in $C([0, T], \mathbb{R})$.

Proof: To simplify the notation, we define $S(s, t, \phi) = \int_s^t k(\phi(u)) du$ for $\phi \in C([s, t], \mathbb{R})$. Assume that solution x_t is uniquely defined on $[0, t_0]$ for $t_0 \in [0, T]$. We consider the following integral equation for $t \in [t_0, T]$

$$\begin{aligned} x_t &= H_1(S(0, t_0, x) + S(t_0, t, x)) + \int_0^{t_0} H_2(S(s, t_0, x) + S(t_0, t, x)) ds \\ &\quad + \int_0^{t_0} H_3(x_s) H_4(S(s, t_0, x) + S(t_0, t, x)) ds + \int_{t_0}^t H_2(S(s, t, x)) ds \\ &\quad + \int_{t_0}^t H_3(x_s) H_4(S(s, t, x)) ds. \end{aligned} \quad (33)$$

Now define a operator $F : C([t_0, T], \mathbb{R}) \rightarrow C([t_0, T], \mathbb{R})$ as follows:

$$\begin{aligned} F(\phi)(t) &= H_1(S(0, t_0, x) + S(t_0, t, \phi)) + \int_0^{t_0} H_2(S(s, t_0, x) + S(t_0, t, \phi)) ds \\ &\quad + \int_0^{t_0} H_3(x_s) H_4(S(s, t_0, x) + S(t_0, t, \phi)) ds + \int_{t_0}^t H_2(S(s, t, \phi)) ds \\ &\quad + \int_{t_0}^t H_3(\phi(s)) H_4(S(s, t, \phi)) ds. \end{aligned}$$

To simplify the notation, we denote the i -th term on the r.h.s. of the above expression by $F_i(\phi)$ for $i = 1, 2, 3, 4, 5$. We denote the supremum (Lipschitz constant) of H_i by $H_{i\infty}(L_i)$ for $i = 1, 2, 3, 4$. Let L_k be the Lipschitz constant of $k(\cdot)$. Then for $\phi^1, \phi^2 \in C([t_0, T], \mathbb{R})$ the followings hold: for $t \in [t_0, T]$

$$\begin{aligned} |F_1(\phi^1)(t) - F_1(\phi^2)(t)| &\leq L_1 L_k(t - t_0)|\phi^1 - \phi^2|_{t_0 t}, \\ |F_2(\phi^1)(t) - F_2(\phi^2)(t)| &\leq L_2 L_k t_0(t - t_0)|\phi^1 - \phi^2|_{t_0 t}, \\ |F_3(\phi^1)(t) - F_3(\phi^2)(t)| &\leq H_{3\infty} L_4 L_k t_0(t - t_0)|\phi^1 - \phi^2|_{t_0 t}, \\ |F_4(\phi^1)(t) - F_4(\phi^2)(t)| &\leq L_2 L_k(t - t_0)^2|\phi^1 - \phi^2|_{t_0 t}, \\ |F_5(\phi^1)(t) - F_5(\phi^2)(t)| &\leq H_{3\infty} L_4 L_k(t - t_0)^2|\phi^1 - \phi^2|_{t_0 t} + H_{4\infty} L_3(t - t_0)|\phi^1 - \phi^2|_{t_0 t}. \end{aligned}$$

Hence combining the above expressions we get, for $t \in [t_0, T]$

$$|F(\phi^1)(t) - F(\phi^2)(t)| \leq (L_1 L_k + L_2 L_k T + H_{3\infty} L_4 L_k T + H_{4\infty} L_3)(t - t_0)|\phi^1 - \phi^2|_{t_0 t}.$$

Hence we can choose $h > 0$ small enough so that $\sup_{[t_0, t]} |F(\phi^1)(s) - F(\phi^2)(s)| < \varrho |\phi^1 - \phi^2|_{t_0 t}$ for some positive $\varrho < 1$ and $t - t_0 = h$. So by contraction mapping theorem, there exists a unique continuous function x defined on $[t_0, t]$ satisfying (33).

Putting $t_0 = 0$, we see that x_t satisfies (32) on $[0, h]$. Having the solution defined on $[0, nh \wedge T]$, we can extend it uniquely on $[0, (n+1)h \wedge T]$ for $n \in \mathbb{N}$. Since $h > 0$ is fixed, this defines the solution uniquely on $[0, T]$. \square .

We can extend Lemma 5.8 as follows:

Lemma 5.9 *Fix $T > 0$. Let $k, H_i : \mathbb{R} \rightarrow \mathbb{R}, i = 1, 2, 3, 4, 5$, be Lipschitz continuous. We also assume H_4, H_5 to be bounded. Then the following integral equation*

$$\begin{aligned} x_t &= H_1(S(0, t)) + \int_0^t H_2(S(s, t))ds + \int_0^t H_3(x_s)H_5(x_s)H_4(S(s, t))ds \\ x_0 &= H_1(0). \end{aligned} \quad (34)$$

has a unique solution in $C([0, T], \mathbb{R})$ where $S(s, t) = \int_s^t k(x_u)du$.

Proof: Let $\varphi_n : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth cut-off function such that $0 \leq \varphi_n \leq 1$, $\varphi_n(s) = 1$ on $[-n, n]$ and $\varphi_n(s) = 0$ outside of $[-n-1, n+1]$. Define $H_i^n(s) = \varphi_n(s)H_i(s)$ for $i = 1, 2, 3$. Then H_i^n is a bounded, Lipschitz continuous function for $i = 1, 2, 3$. Hence $H_3^n H_5$ is a bounded, Lipschitz continuous function for $n \in \mathbb{N}$. Therefore applying Lemma 5.8, we have unique $x^n : [0, T] \rightarrow \mathbb{R}$, continuous, satisfying

$$\begin{aligned} x_t^n &= H_1^n\left(\int_0^t k(x_s^n)ds\right) + \int_0^t H_2^n\left(\int_s^t k(x_u^n)du\right)ds \\ &\quad + \int_0^t H_3^n(x_s^n)H_5(x_s^n)H_4\left(\int_s^t k(x_u^n)du\right)ds \\ x_0^n &= x_0 = H_1(0). \end{aligned} \quad (35)$$

Since $k, H_i, i = 1, \dots, 5$ are Lipschitz and H_4, H_5 are bounded, we can get positive constants d_1, d_2 such that

$$\begin{aligned} |H_1^n(\int_0^t k(x_s^n)ds)| &\leq d_1 + d_2 \int_0^t |x_s^n|ds \\ |\int_0^t H_3^n(x_s^n)H_5(x_s^n)H_4(\int_s^t k(x_u^n)du)ds| &\leq d_1 + d_2 \int_0^t |x_s^n|ds, \end{aligned}$$

and

$$|\int_0^t H_2^n(\int_s^t k(x_u^n)du)ds| \leq d_1 + d_2 \int_0^t \int_s^t |x_u^n|duds \leq d_1 + d_2 T \int_0^t |x_s^n|ds,$$

for all $t \in [0, T]$ and $n \in \mathbb{N}$. Now combining these estimates with (35) and applying Gronwall's inequality we have

$$\sup_n \sup_{[0, T]} |x_s^n| \leq d_3, \quad (36)$$

for some constant d_3 . For any compact $C \subset \mathbb{R}$, there exists constant L_C such that $|H_i^n(x) - H_i^n(y)| \leq L_C|x - y|$ for all $x, y \in C$ and $n \in \mathbb{N}$. Therefore using an analogous expression as (33), we get a constant $d_4 > 0$ (depending on d_3) satisfying

$$\sup_n \sup_{0 \leq s \leq t \leq T} |x_s^n - x_t^n| \leq d_4|t - s|. \quad (37)$$

(36) and (37) imply that the sequence $\{x^n\}$ is equi-continuous family of continuous functions on $[0, T]$. Therefore using Arzelà-Ascoli, theorem there is a $x : [0, T] \rightarrow \mathbb{R}$, continuous, such that $|x^{n_k} - x|_{0T} \rightarrow 0$ along some subsequence $n_k \rightarrow \infty$. Hence letting $n_k \rightarrow \infty$ in (35) we have

$$\begin{aligned} x_t &= H_1(\int_0^t k(x_s)ds) + \int_0^t H_2(\int_s^t k(x_u)du)ds \\ &\quad + \int_0^t H_3(x_s)H_5(x_s)H_4(\int_s^t k(x_u)du)ds \\ x_0 &= x_0 = H_1(0). \end{aligned}$$

This proves the existence of solution to (34). To prove uniqueness, let \bar{x} be another solution to (34). Define $\sigma_n = \inf\{t \geq 0 : |\bar{x}_t| > n\} \wedge T$. Since $H^n(s) = H(s)$ for $|s| \leq n$, from Lemma 5.8, we get $x_s = x_s^n = \bar{x}_s$ for $s \leq \sigma_n$ and for all n large (we need to take large n to ensure that $x_0 \in [-n, n]$). Therefore to complete the proof it is enough to show that $\liminf_{n \rightarrow \infty} \sigma_n = T$. But this is obvious as $\sup_{[0, T]} |\bar{x}_s| < \infty$ (follows from a simple calculation similar to (36)). \square

5.2 Appendix B

Consider a sequence of probability measures $\{\nu^n\}$ and ν on $[0, \infty)$ such that $\nu^n \rightarrow \nu$ as $n \rightarrow \infty$. Let $\{v_i^n\}_{i=-\infty}^{i=\infty}$ be an i.i.d. sequence with common probability distribution ν^n . For $m \in \mathbb{Z}$ and $\ell \geq 0$, define

$$\mathcal{L}^n(m, \ell) = \frac{1}{n} \sum_{i=1+m}^{m+\lfloor n\ell \rfloor} \delta_{v_i^n}. \quad (B1)$$

By Skorohod representation theorem, there exists $[0, \infty)$ -valued random variables $Y^n \sim \nu^n$ and $Y \sim \nu$ such that $Y^n \rightarrow Y$ almost surely on some common probability space. Define $\bar{Y} = \sup_n Y^n$. Let $\bar{\nu}$ be the law of \bar{Y} . There exists a continuous, increasing, unbounded function \bar{f} such that $\bar{f} \geq 1$ and $\langle \bar{f}^2, \bar{\nu} \rangle < \infty$ (see Appendix B in [18]). Define

$$\mathcal{V} = \{\chi_{C_x}, x \geq 0\} \cup \{\chi_{\bar{C}_x}, x \geq 0\} \cup \{\bar{f}\}.$$

Lemma 5.10 *Fix $M, L > 0$. Under the above assumptions, for all $\epsilon, \eta > 0$ we have*

$$\limsup_{n \rightarrow \infty} \mathbb{P}_n \left(\max_{-nM < m < nM} \sup_{\ell \in [0, L]} \sup_{f \in \mathcal{V}} |\langle f, \mathcal{L}^n(m, \ell) \rangle - \ell \langle f, \nu^n \rangle| > \epsilon \right) < \eta.$$

For the proof of above lemma we refer Lemma B.1 in [18]. Following the same argument as in Lemma 5.1 in [18] we can have a sequence $\epsilon_A(n)$ such that $\epsilon_A(n) \rightarrow 0$ as $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} \mathbb{P}_n(\Omega_A^n(M, L)) = 1$ for every fixed $M, L > 0$ where

$$\Omega_A^n(M, L) = \left\{ \max_{-nM < m < nM} \sup_{\ell \in [0, L]} \sup_{f \in \mathcal{V}} |\langle f, \mathcal{L}^n(m, \ell) \rangle - \ell \langle f, \nu^n \rangle| \leq \epsilon_A(n) \right\}. \quad (\text{B2})$$

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